

*Ursus arctos horribilis*: dynamic modeling of Canadian population© 2019. I. Bica<sup>1</sup> ORCID: 0000-0003-0791-3016, M. Solomonovich<sup>1</sup> ORCID: 0000-0002-3210-2904K. Deutscher<sup>2</sup> ORCID: 0000-0003-4323-3915, A. Garrett<sup>1</sup> ORCID: 0000-0002-8111-3407K. Burak<sup>3</sup> ORCID: 0000-0001-7119-7777, H. Peacock<sup>1</sup> ORCID: 0000-0001-9657-1235<sup>1</sup>MacEwan University,

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The grizzly bears are  $K$ -strategists and their innate tendency is to reach homeostasis. In the First Nations folklore grizzly bears are viewed as “spirits” that bring balance in their untamed habitat where they roam, this being an indication that they do not overpopulate their habitat and their gene flow is “designed” to reach homeostasis without surpassing it. In the present article we study the dynamics of the grizzly bear population in the Southwest Alberta, Canada. Based on the dynamical model with three parameters, we obtain estimates for the carrying capacity and the minimum viable population of the grizzly bear population in their dynamical habitat. The article starts with the discussion of the rationale for choosing the Logistic Growth Model as the most appropriate for describing the dynamics of grizzly population. In addition to the usual for this kind of models parameters of the growth rate and the carrying capacity, in the current model we consider the parameters of Minimum Viable Population ( $MVP$ ) and Safe Harbour ( $SH$ ) – a measurement introduced by the Alberta Grizzly Bear Recovery Plan. The first of these parameters ( $MVP$ ) is determined by the essential number of the individuals that would allow the survival of the species. The latter measurement ( $SH$ ) is related to the so-called Grizzly Bear Priority Areas, where the risk of mortality is low. Then, based on Verhulst model and Statistical data, the carrying capacity and growth rate for the female grizzly bears in Alberta have been obtained. Mathematical analysis of the model has shown that the equilibria at  $K$  (carrying capacity) and  $MVP \cdot SH$  are, respectively, stable and unstable. The time of possible extinction for the populations with the initial conditions below the threshold  $MVP \cdot SH$  has been numerically estimated. The correlation between the system parameters and its influence on the survival of the population has been analyzed and the recommendations on ensuring the survival have been given.

**Keywords:** grizzly bears, *Ursus arctos horribilis*, population dynamics, mathematic modeling, Southwest Alberta, Canada.

УДК 51.76

**Медведь гризли (*Ursus arctos horribilis*):  
математическое моделирование канадской популяции**

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Стратегия размножения североамериканских медведей гризли относится к  $K$ -типу, и их врождённая тенденция – достигать гомеостаза. В фольклоре коренных жителей северной Америки – индейцев медведи гризли рассматриваются как «духи», которые приводят в баланс дикую среду обитания, где они свободно бродят, не перенасыщая свой ареал, и они генетически предрасположены к достижению гомеостаза, не допуская перенаселения популяции. В настоящей статье построена модель динамики популяции гризли в юго-западной части провинции Альберта

(Канада), дана оценка ёмкости среды ( $K$ ) и минимальной жизнеспособной численности популяции гризли в их динамичной среде обитания. Приведена аргументация в пользу выбора логистической модели роста как наиболее подходящей для описания динамики численности популяции медведей гризли. В дополнение к обычному набору параметров, присущему таким моделям – скорости роста численности популяции и ёмкости среды, в данную модель включены также параметры «минимальная жизнеспособная численность популяции» ( $MVP$ ) и «безопасная зона» ( $SH$ ). Последний параметр был введён Программой по восстановлению численности медведей гризли в провинции Альберта. Параметр  $MVP$  определяется минимальным числом особей, необходимым для выживания популяции. Параметр  $SH$  приписывается так называемым «приоритетным областям проживания медведей гризли», где условия для их проживания благоприятны, а смертность невысока. Далее, основываясь на модели Ферхюльста и статистических данных, дана оценка ёмкости среды и скорости роста численности популяции для медведиц гризли в Альберте. Математический анализ модели показывает, что равновесие системы при значении численности, стремящемся к ёмкости среды  $K$  – устойчивое, в то время как равновесие около значения  $MVP \cdot SH$  – неустойчивое. Получена численная оценка для времени вымирания популяции с начальными условиями ниже этого значения. Проанализированы корреляции между параметрами и их влияние на выживание популяции медведей. На основании анализа даны рекомендации по обеспечению выживания популяции.

**Ключевые слова:** медведи гризли, *Ursus arctos horribilis*, динамика популяции, математическое моделирование, юго-западная Альберта, Канада.

The grizzly bears are K-strategists and their innate tendency is to reach homeostasis. In the First Nations folklore grizzly bears are viewed as “spirits” that bring balance in their untamed habitat where they roam, this being an indication that they do not overpopulate their habitat and their gene flow is “designed” to reach homeostasis without surpassing it [1]. In the present article we study the dynamics of the grizzly bear population in the Southwest Alberta, Canada, giving estimates for the carrying capacity and the minimum viable population of the grizzly bear population in their dynamical habitat.

The origins of the logistic model goes all the way back to 1798, when the Reverend Thomas Robert Malthus (alias Joseph Johnson) published An Essay on the Principle of Population [2]. His essay opened the idea that populations of any origins and species can be measured mathematically. In his own words he said: “I said that population, when unchecked, increased in a geometrical ratio, and subsistence for man in an arithmetical ratio. Let us examine whether this position be just...”.

The Malthusian growth is known as:

1.  $P_t = P_0 r^t$ , geometric population growth, when the population size is modeled in discrete time intervals, and organisms have non-overlapping generations.  $r > 0$ , all the time constant, is the net reproductive rate & intergenerational rate of change (i. e. the geometric rate of increase),  $P_0$  is the initial population, and  $P_t$  is the population size at time  $t$  where the time  $t$  is measured in time units or generations.

2.  $P(t) = P_0 e^{rt}$ , exponential population growth, when the population growth depends on conditions at the current moment, and the population growth is continuous (continuous reproduction/overlapping generations).  $r > 0$

represents the instantaneous rate of increase,  $P_0$  is the initial population, and  $P(t)$  is the population size at time  $t$  where the time  $t$  is measured continuously (i. e.  $t \geq 0$ ). The geometric growth is the discrete form of the exponential growth. The exponential growth is the solution of the elementary ODE model

$$\frac{dP}{dt} = rP .$$

The Malthusian model is appropriate for density-independent populations, while the grizzly bear population shows density-dependent feedback, i. e.:

- From all historical data, the bear population never showed an indefinite exponential increase.
- Resources for the bear population are limited, influencing birth and death.
- From historic data, the bear population growth rate showed decline.

Therefore the Malthusian model is not adequate for such population. An adequate model for studying the grizzly bear population is the Verhulst's model, the classic Logistic Growth Model:

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right), \quad (1)$$

where  $K$  is the carrying capacity (a function of supply of limiting resources), its function being to assure a sustainable population size for the grizzlies based on prevailing environmental conditions. We have population growth positive below  $K$ , and negative above  $K$ . The thresholds of the Verhulst model are  $P = 0$  and  $P = K$ . In the Figure 1 it is shown an example of the Verhulst's model with carrying capacity  $K = 1000$ .

We notice the negative growth rate of the population if the initial population is above the threshold  $P = K$ , and the positive growth rate

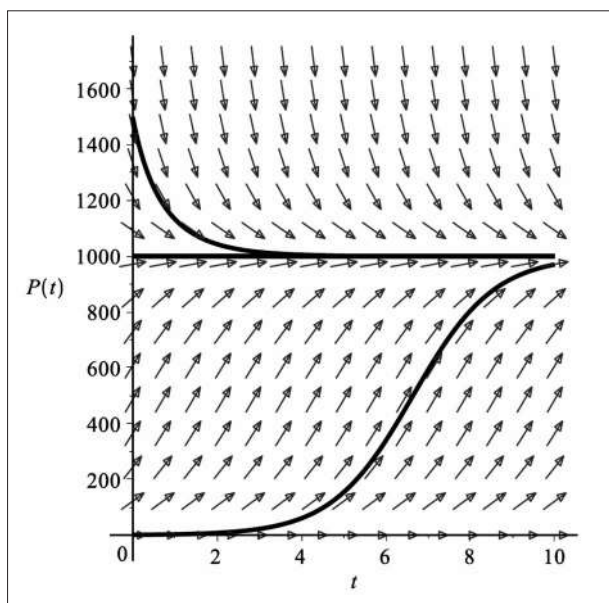


Fig. 1. The classic Verhulst's model for carrying capacity  $K = 1000$

if the initial population is below the threshold  $P = K$ . Biologically the threshold  $P = 0$  is not realistic. The model that we consider for studying the dynamics of the grizzly bear population, which represents the motivation of the present article, introduces an Allee type effect, the threshold  $P = SH \cdot MVP$  by replacing the threshold  $P = 0$  as follows:

$$\frac{dP}{dt} = r \left( \frac{P}{SH \cdot MVP} - 1 \right) \left( 1 - \frac{P}{K} \right) \quad (2)$$

$SH$  represents the **Safe Harbour**, a new measuring function introduced by the Alberta Grizzly Bear Recovery Plan 2008–2013 [3], which speaks of the creation of **Grizzly Bear Priority Areas** in high quality habitat where there is a low risk of mortality; the Safe Harbour ( $SH$ ) is a combination of high quality habitat and reduced risk. The  $MVP$  is referred to the **Minimum Viable Population**, a measure that specifies the necessary amount of individuals in order for the species to survive. While the threshold  $P = K$  is a stable equilibrium point of the dynamical system, the threshold  $P = SH \cdot MVP$  is an unstable equilibrium point of the system.  $SH$  acts as a buffer zone for the  $MVP$ ; it allows it to be depressed and then to rebound. Using  $SH$  as a buffer of the  $MVP$  we are able to extract, using an inverse problem, an estimate for the  $MVP$ . What is very important to be realized is that the model (2), when the population is situated under the  $SH \cdot MVP$ -threshold it is biologically relevant only up to the time of extinction! Once the popu-

lation is extinct, the relevance of a population dynamics model is no longer necessary, as it is irrelevant. We believe that we bring something new in the field of Conservation Biology, as until now the  $MVP$  is only estimated statistically by using computer simulations for Population Viability Analysis (PVA), some important resources in this direction being [4] and [5].

### Carrying capacity and female growth rate estimates for grizzly bears in the province of Alberta, Canada

The grizzly bear is a flagship/umbrella species that occupies an estimated area of  $2.98 \cdot 10^6 \text{ km}^2$  in Canada [6]. They are habitat generalists, and they can be found from sea level to high alpine, temperate coastal rain forest, alpine tundra, mountain slopes, upland boreal forest, taiga, grasslands, and tundra. In Alberta there are seven Bear Management Areas (BMA): BMA 1 – Chinchaga (boreal grizzly habitat in Alberta), BMA 2 – Grande Cache, BMA 3 – Yellowhead, BMA 4 – Clearwater, BMA 5 – Livingstone, BMA 6 – Waterton-Castle, BMA 7 – Swan Hills. Our assumption is that grizzly bears have a dynamic habitat (no fixed boundaries / home range with extensions). One reason supporting this assumption is the resource pressure that forces micro-migration. Then the carrying capacity becomes a dynamic variable, which depends on the density of natural resources of the particular habitat frequented by the grizzly bear during a certain time of the year (e. g. salmon feeding). The assumption of dynamical habitat for the grizzly bear supports the known fact that the grizzly bear is an umbrella species (influences the survival of other species). Estimating the carrying capacity of the grizzly bear in their dynamical habitat can make us understand better the limitations of the home range habitat for the grizzly bear. Figure 2 (see color insert) shows the distribution of grizzly bears in the BMA 2–7 (the areas on which we focused our present work) outlining the home range and shared habitat for grizzlies with the estimated population numbers as specified in the document prepared for Alberta Sustainable Resource Development (ASRD), Alberta Conservation Association (ACA) [7].

The grizzly bears have a slow reproductive rate due to the fact that the female grizzly bears have a late reproductive maturity, combined with small litter and long inter-litter intervals [8, 9].

To obtain an estimate for the carrying capacity and for the female growth rate we are using the Verhulst model (1); the carrying capacity

estimate will be for the female grizzly bears, then we will adjust it to the overall population of grizzly bears male & female. The solution of the model (1) with the initial condition  $P(0) = P_0$  can be found by elementary integration to be:

$$P(t) = \frac{KP_0}{e^{-rt}(K - P_0) + P_0} \quad (3)$$

We are interested to find  $K, r$  and  $P_0$  knowing the total number of female population at three equally-spaced times  $d, 2d$  and  $3d$  respectively, where  $d$  is our time unit. We obtain then the following system of nonlinear equations:

$$\frac{KP_0}{e^{-ird}(K - P_0) + P_0} = P(id), \quad i=1,2,3 \dots \quad (4)$$

which transforms into the system:

$$e^{-ird}(x_0 - M) = P_i - M, \quad i = 1 \dots 3, \quad (5)$$

where:

$$x_0 = 1/P_0, M = 1/K, P_i = 1/P(id), i = 1 \dots 3. \quad (6)$$

The system (5) provides the exact solution:

$$M = P_1 - C, \quad C = \frac{P_1 - P_2}{1 - x}, \quad x = \frac{P_2 - P_3}{P_1 - P_2},$$

$$r = -\frac{1}{d} \ln x,$$

$$x_0 = P_1 + \frac{P_1 - P_2}{x}.$$

Using (6) we get the exact solution of the system (4):

$$K = \frac{1}{P_1 - C}, \quad C = \frac{P_1 - P_2}{1 - x}, \quad x = \frac{P_2 - P_3}{P_1 - P_2}, \quad (7)$$

$$r = -\frac{1}{d} \ln x, \quad (8)$$

$$P_0 = \frac{1}{P_1 + \frac{P_1 - P_2}{x}}. \quad (9)$$

We used the following data for the female grizzly bear population [10–11]:

|                                  |      |      |      |      |      |      |
|----------------------------------|------|------|------|------|------|------|
| Year                             | 2008 | 2009 | 2010 | 2011 | 2012 | 2013 |
| Female population size estimates | 441  | 489  | 539  | 588  | 636  | 677  |

The first estimates for  $K, r$  and  $P_0$  were obtained by considering the three consecutive years 2009, 2010 and 2011. The value obtained for  $P_0$  was compared with the known data for the female population size estimate from the previous year, i.e. the year 2008. Then we used two more sequences of consecutive years to find estimates for  $K, r$  and  $P_0$ : 2010, 2011, 2012 and 2011, 2012, 2013, as this was the only available data that we could find. In the Population Size and Trends Section in the COSEWIC Assessment and Status Report on the Grizzly Bear *Ursus arctos* in Canada – 2012 [6] it is explained how difficult it is to obtain such data, and yet the limited data that we found allowed us to

show that the method used has a great potential in the population ecology/computational biology research fields. We expected that the values obtained for  $P_0$  for each trial of three consecutive years to be close to the estimate of the female population size from the year prior to the first year in the sequence of the three consecutive years, which it did happen. Here are our estimates; in each table we start with the year prior to the first year (**in bold font**) in the sequence of the three consecutive years (the three consecutive years are *in italic font*) that we used in order to compare the estimate obtained for  $P_0$  (the estimate found for  $P_0$  is **in bold font** as well), as explained above.

|            |            |            |            |     |      |            |
|------------|------------|------------|------------|-----|------|------------|
| 2008       | 2009       | 2010       | 2011       | $K$ | $r$  | $P_0$      |
| <b>441</b> | <i>489</i> | <i>539</i> | <i>588</i> | 980 | 0.20 | <b>439</b> |

|            |            |            |            |      |      |            |
|------------|------------|------------|------------|------|------|------------|
| 2009       | 2010       | 2011       | 2012       | $K$  | $r$  | $P_0$      |
| <b>489</b> | <i>539</i> | <i>588</i> | <i>636</i> | 1058 | 0.19 | <b>489</b> |

|            |            |            |            |     |      |            |
|------------|------------|------------|------------|-----|------|------------|
| 2010       | 2011       | 2012       | 2013       | $K$ | $r$  | $P_0$      |
| <b>539</b> | <i>588</i> | <i>636</i> | <i>677</i> | 830 | 0.30 | <b>533</b> |

We averaged the values for  $K$  and  $r$  as shown in the tables below



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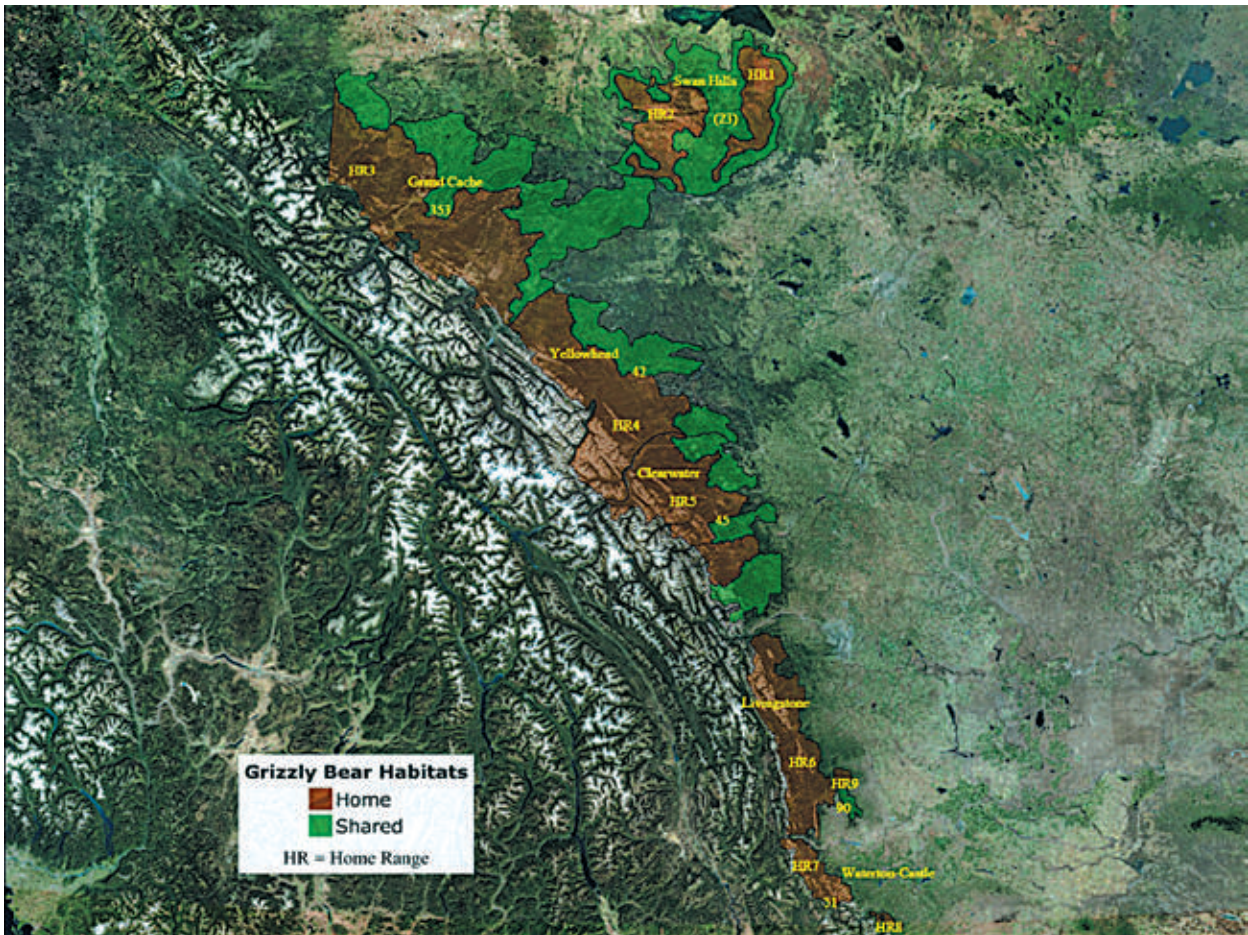


Fig. 2. BMA 2-7. Home range and shared habitat

|  |                  |
|--|------------------|
| $Kp$ = carrying capacity for the entire grizzly population, male & female<br>The mean of the three estimates of $K$ for females – 100/64 | Estimate<br>1493 |
| $r$ = female reproductive rate<br>The mean of the three estimates of $r$   | Estimate<br>0.23 |

The value of  $r = 0.23$  for the female reproductive rate is in agreement with [9]. The division by 64 in estimating  $Kp$  is due to the inclusion of the male population as well. Using the data provided by [9] we have for 690 bears: Female Demographics – 52 cubs, 74 yearlings, 83 sub-adults, 227 adults, total female population size = 436; total male population size 690 – 436 = 254. Male and female ratio of the population for all ages: 36% males and 64% females.

The ratio found above is very comparable with the one found by [11] much earlier in 1994 for grizzly bears in the Northwest Territories (a study done between 1987 and 1989); 33% males and 67% females. We would have gotten the same ratio if we would have truncated off the female proportion first.

### Mathematical analysis of the new proposed model

As we mentioned in Section 1, biologically the threshold  $P = 0$  is not realistic. The model that we consider for studying the dynamics of the grizzly bear population is the model (2), which introduces the Allee type effect given by the new threshold  $P = SH \cdot MVP$ , replacing the threshold  $P = 0$  in the Verhulst model (1):

$$\frac{dP}{dt} = r \left( \frac{P}{SH \cdot MVP} - 1 \right) \left( 1 - \frac{P}{K} \right), \quad (10)$$

where  $SH$  represents the Safe Harbour that refers to the Grizzly Bear Priority Areas, and  $MVP$  is the Minimum Viable Population, the measure that specifies the necessary amount of individuals in order for the species to survive. While the threshold  $P = K$  is a stable equilibrium point of the dynamical system, the threshold  $P = SH \cdot MVP$  is an unstable equilibrium point of the system.

**Safe Harbour for the Grizzly Habitat.** In 2009 the Alberta Sustainable Resource Development Directors Council officially approved the designation of Grizzly Bear Core Areas and Secondary Areas. A Core Area or a Secondary Area for a grizzly bear is called Grizzly Bear Watershed Unit (GBWU), a unit which represents a subdivision of major watersheds to

approximate the home range of a female grizzly bear, which is 700 km<sup>2</sup> [12]. The Core Areas are prime areas for grizzlies with a high resource factor that promotes their survival, and where the mortality risk is low; the mortality risk is currently measured through open road densities. The Secondary Areas reflect the broader range of grizzly bears, and are considered good grizzly bear habitat areas [12]. The quality of a grizzly bear habitat is measured in term of:

- The interaction with their biotic environment; a good habitat will offer a balanced interaction of grizzlies with their biotic and abiotic environment, which will be measured by an optimum resource availability for grizzlies while not depleting the resource availability of the biotic environment they reside in (they are an umbrella species). The resource availability is modeled as Resource Selection Function (RSF), which is the relative probability of grizzly bear occurrence on the landscape.

- The safety of the habitat; a good habitat will offer an optimum security for accessing resources (resource availability) that will encourage reproduction, and it will provide a safe environment to raise offsprings. The safety of the habitat is related to the security for the grizzlies, which is related to human-caused mortality. The human-caused mortality is associated to Open Road Density (the sum of all human- infrastructure within the habitat), which through the suggestion of the Grizzly Bear Recovery Team is eventually replaced with Mortality Risk, quote from [12]: “Open Road Density is eventually replaced with Mortality Risk. Mortality Risk is a spatial model that represents the relative probability of human-caused grizzly bear mortality. It is a function of terrain ruggedness, distance from roads, streams, cutlines, and forest edges and land status (protected area, Green/White Area). For the purpose of this analysis, both Open Road Density and Mortality Risk will be used. Safe Harbour is a combination of habitat quality and risk. A safe harbour is an area of good habitat (high RSF values), to which bears are attracted by an abundance of resources, but also where the bear faces a low risk of human caused

mortality (low Mortality Risk). Safe harbour was calculated using the following expression:

$$SF = RSF \cdot (10 - RISK)''.$$

– The connectivity of the habitat; a good habitat will assure access to other resources outside the home range, a home range with extensions.

On the stability of equilibrium points  $P = K$  and  $P = SH \cdot MVP$ . Proving that:

$$\lim_{P \rightarrow K} \frac{dP}{dt} = 0, \tag{11}$$

let  $\varepsilon > 0$  and choose  $|P - K| < \delta_\varepsilon$ , with:

$$\delta_\varepsilon = \frac{-r \left[ \left( \frac{K}{SH \cdot MVP} - 1 \right) + \sqrt{r^2 \left( \frac{K}{SH \cdot MVP} - 1 \right)^2 + \frac{4rK\varepsilon}{SH \cdot MVP}} \right]}{2r / (SH \cdot MVP)} > 0$$

Then the new model (10) will give us the following:

$$\left| \frac{dP}{dt} \right| \leq \frac{r}{K \cdot SH \cdot MVP} |P - K|^2 + \frac{r}{K} \left| \frac{K}{SH \cdot MVP} - 1 \right| |P - K| < \frac{r}{K \cdot SH \cdot MVP} \delta_\varepsilon^2 + \frac{r}{K} \left| \frac{K}{SH \cdot MVP} - 1 \right| \delta_\varepsilon = \varepsilon \Rightarrow \lim_{P \rightarrow K} \frac{dP}{dt} = 0$$

Similarly it can be proven that:

$$\lim_{P \rightarrow SH \cdot MVP} \frac{dP}{dt} = 0. \tag{12}$$

Figure 3 shows the way the slopes of the tangents to the integral curves of the model (10)

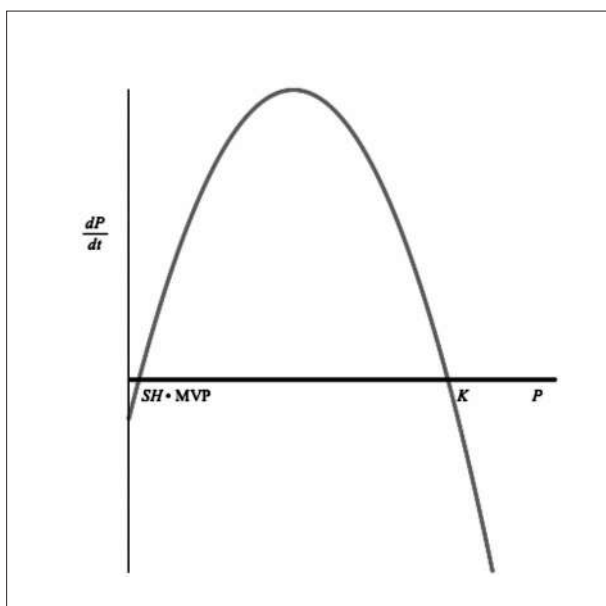


Fig. 3.  $dP/dt$  vs.  $P$  in the model (10)

vary with respect to  $P(t)$  for some prescribed values of the parameters. The elementary analysis of the right hand side quadratic in  $P(t)$  in the model (10) gives us the following:

– In the vicinity of  $P = SH \cdot MVP$  we have:

$$\frac{dP}{dt} < 0, \text{ if } P < SH \cdot MVP, \tag{13}$$

$$\text{and } \frac{dP}{dt} > 0, \text{ if } P > SH \cdot MVP.$$

– In the vicinity of  $P = K$  we have:

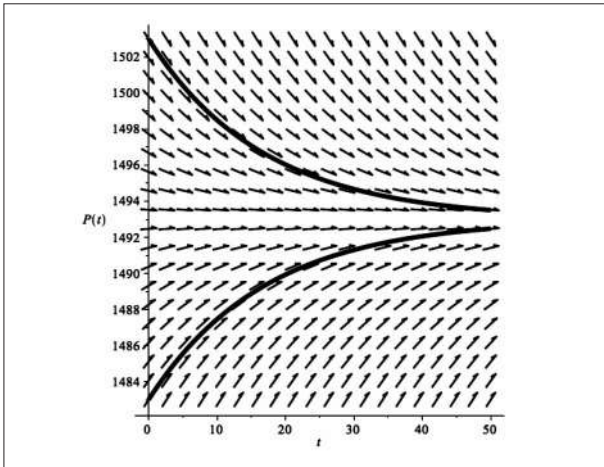
$$\frac{dP}{dt} > 0, \text{ if } P < K, \tag{14}$$

$$\text{and } \frac{dP}{dt} < 0, \text{ if } P > K.$$

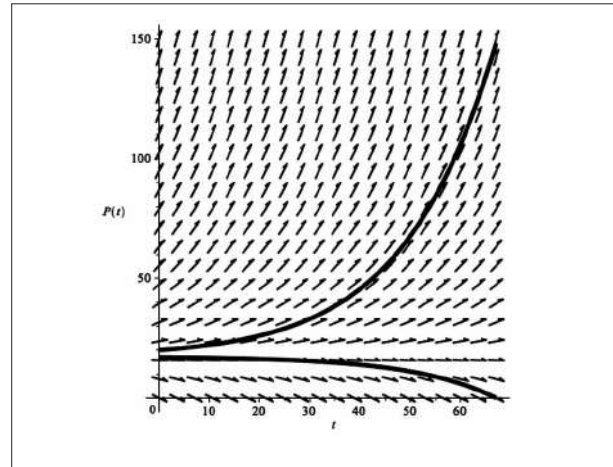
The relationships (11) and (14) as well as (12) and (13) prove that the equilibrium points  $P = K$  and  $P = SH \cdot MVP$  of the model (10) are stable and unstable, respectively.

If the the density of the population is located under the threshold  $P = SH \cdot MVP$ , then the time of extinction for the population is obtained from the model (10) to be:





**Fig. 4.**  $P = K$  is a stable equilibrium point of the model (10). The following values were considered for parameters:  $K = 1493$ ,  $r = 1.04$ ,  $MVP = 40$ ,  $SH = 0.433$ . The value of  $r$  was taken from [9], the value of  $SH$  was taken from [12], and we used our estimate that we found for  $K = \hat{K}_p$



**Fig. 6.**  $P = SH \cdot MVP$  is an unstable equilibrium point of the model (10). The following values were considered for parameters:  $K = 1493$ ,  $r = 1.04$ ,  $MVP = 40$ ,  $SH = 0.433$ ,  $P_0 = 17$ . The value of  $r$  was taken from [9], the value of  $SH$  was taken from [12], and we used our estimate that we found for  $K = \hat{K}_p$ . The estimated time of extinction evaluated, using the formula (15):  $T_{ext} \approx 67.06$ .

$$T_{ext} = \frac{SH \cdot MVP \cdot K \cdot \ln \left( \frac{SH \cdot MVP \cdot (K - P_0)}{K(SH \cdot MVP - P_0)} \right)}{r(K - SH \cdot MVP)}, \quad (15)$$

where  $P_0$  represents the initial population given by the initial condition  $P(0) = P_0$ .

Figures 4 and 5 show the stability of the thresholds  $P = SH \cdot MVP$  and  $P = K$  respectively;  $P = SH \cdot MVP$  is unstable and  $P = K$  is stable. For the threshold  $P = SH \cdot MVP$  the time span used is considered only until extinction (i. e.

$t \in [0; T_{ext}]$ ), where the model (10) is biologically relevant.

**Initial Value Problem for the model (10).** Using the model (10) we create the Initial Value Problem (IVP):

$$\frac{dP}{dt} = r \left( \frac{P}{SH \cdot MVP} - 1 \right) \left( 1 - \frac{P}{K} \right), \quad (16)$$

$$P(0) = P_0. \quad (17)$$

The IVP (16–17) has the unique solution:

$$P(t) = \left[ SH \cdot MVP \left( K \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP(K - P_0)} \right)^{\frac{K}{K - SH \cdot MVP}} \cdot e^{\frac{rt}{SH \cdot MVP}} + \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP(K - P_0)} \right)^{\frac{SH \cdot MVP}{K - SH \cdot MVP}} \cdot e^{\frac{rt}{K}} \right) \right] / \left[ \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP(K - P_0)} \right)^{\frac{SH \cdot MVP}{K - SH \cdot MVP}} \cdot e^{\frac{rt}{K}} + SH \cdot MVP \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP(K - P_0)} \right)^{\frac{K}{K - SH \cdot MVP}} \cdot e^{\frac{rt}{SH \cdot MVP}} \right]. \quad (18)$$

**Fitness – MVP Correlation.** In this article, by fitness we understand simply the total number of offspring in a chosen time unit. We define the fitness function to be:

$$F(MVP, SH, t) = P(t) - P_0, \quad (19)$$

where  $P(t)$  is given by (18), and  $P_0$  represents the initial population size in a BMA. Simplifying (19) we obtain:



$$\left( SH \cdot MVP \cdot (K - P_0) \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP \cdot (K - P_0)} \right)^{\frac{K}{K - SH \cdot MVP}} e^{\frac{rt}{SH \cdot MVP}} + (SH \cdot MVP - P_0) \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP \cdot (K - P_0)} \right)^{\frac{SH \cdot MVP}{K - SH \cdot MVP}} e^{\frac{tr}{K}} \right) / \left( \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP \cdot (K - P_0)} \right)^{\frac{SH \cdot MVP}{K - SH \cdot MVP}} e^{\frac{tr}{K}} + SH \cdot MVP \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP \cdot (K - P_0)} \right)^{\frac{K}{K - SH \cdot MVP}} e^{\frac{rt}{SH \cdot MVP}} \right). \quad (20)$$

The model (20) correlates the fitness with the *MVP*, i. e. the model gives an estimate of the *MVP* based on an observed fitness in one year (our time unit of choice) in a BMA. In general the model is designed to give predictions of the *MVP* based on an estimated fitness for a density-dependent population under observation. Considering the equation  $F(MVP, SH, t) = EF$ , the value of the *MVP* obtained by solving the equation tells us what population size is necessary for the grizzly bear to continue surviving when the fitness in one year in the BMA is *EF*.

For *SH* and *t* given, the function  $F(MVP, SH, t)$  is a single variable function, say  $f(MVP)$ . We want to see whether there exists a unique *MVP* such that  $f(MVP) = EF$  when *MVP* is part of a certain open interval *U*. For

the set of data that we used, Grande Cache BMA; core and secondary areas [12], numerically we could show that for *MVP* in the open interval  $U = (0, 811)$  (more than enough realistic interval for *MVP*),  $f'(MVP) \neq 0$  on *U* (Fig. 6). Then from the Inverse Function Theorem there exists an open interval *V* containing *EF* (realistically *EF* is well contained in between  $f(1) \approx 801.35$  and  $f(810) \approx 0.0046$ ) such that *f* maps *U* onto *V* in a one-to-one correspondence. Then there will be a unique *MVP* in *U* such that  $f(MVP) = EF$ .

For the general case, working under the hypothesis that  $P_0$  is located in between the thresholds  $P = SH \cdot MVP$  and  $P = K$ , i. e.  $S \cdot MVP < P_0 < K$ , we can prove that  $f'(MVP) \neq 0$ . Indeed we have:

$$f'(MVP) = \left( \left( (P_0 - SH \cdot MVP)(K - SH \cdot MVP)rt + SH^2 \cdot MVP^2 (K - P_0) \right) \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP (K - P_0)} \right)^{\frac{K + SH \cdot MVP}{K - SH \cdot MVP}} \cdot e^{\frac{rt(SH \cdot MVP - K)}{SH \cdot MVP \cdot K}} + SH \cdot MVP (SH \cdot MVP - P_0) \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP (K - P_0)} \right)^{\frac{2SH \cdot MVP}{K - SH \cdot MVP}} e^{\frac{2rt}{K}} \right) / \left( \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP (K - P_0)} \right)^{\frac{SH \cdot MVP}{K - SH \cdot MVP}} \cdot e^{\frac{rt}{K}} + SH \cdot MVP \left( \frac{P_0 - SH \cdot MVP}{SH \cdot MVP (K - P_0)} \right)^{\frac{K}{K - SH \cdot MVP}} \cdot e^{\frac{rt}{SH \cdot MVP}} \right)^2. \quad (21)$$

Assuming that  $f'(MVP) = 0$ , one can get the following:

$$\left( \frac{(P_0 - SH \cdot MVP)(K - SH \cdot MVP)}{SH^2 \cdot MVP^2 (K - P_0)} + 1 \right) e^{rt \left( \frac{1}{SH \cdot MVP} - \frac{1}{K} \right)} = 1, \quad (22)$$

which is impossible due to the fact that:

$$\frac{(P_0 - SH \cdot MVP)(K - SH \cdot MVP)}{SH^2 \cdot MVP^2 (K - P_0)} + 1 > 1,$$

for any  $MVP$  in any open set, as  $SH \cdot MVP < P_0 < K$ , the hypothesis we work under, and:

$$e^{r \left( \frac{1}{SH \cdot MVP} - \frac{1}{K} \right)} = e^{r \frac{K - SH \cdot MVP}{SH \cdot MVP \cdot K}} > e^0.$$

Hence  $f'(MVP) \neq 0$  for any  $MVP$  in any open set, which assures us that the Inverse Function Theorem, to extract  $MVP$ , will work all the time!

Using the data for the Grande Cache Core area [12], we obtained the following estimates for  $MVP$ :

|                               |     |     |     |     |     |    |    |    |    |    |
|-------------------------------|-----|-----|-----|-----|-----|----|----|----|----|----|
| Fitness for time unit $t = 1$ | 1   | 2   | 3   | 4   | 5   | 6  | 7  | 8  | 9  | 10 |
| $MVP$ estimate                | 361 | 232 | 171 | 135 | 112 | 95 | 83 | 74 | 66 | 60 |

Using the data for the Grande Cache Secondary area [12], we obtained the following estimates for  $MVP$ :

|                               |     |     |     |     |     |     |     |     |     |    |
|-------------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|----|
| Fitness for time unit $t = 1$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10 |
| $MVP$ estimate                | 548 | 352 | 259 | 205 | 170 | 145 | 126 | 112 | 101 | 91 |

The results obtained are not contradictory:

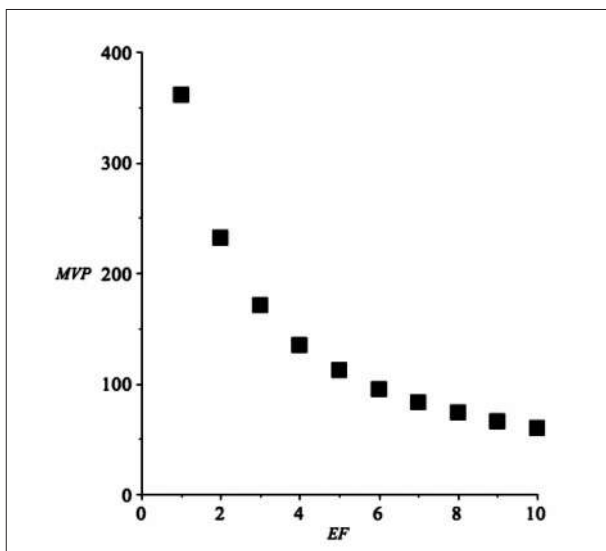
– The fitness and  $MVP$  are correlated by an inverse process; a depression in the fitness triggers a larger requirement for  $MVP$ , which it would be expected. One important factor that can trigger a depression in the fitness may be strongly correlated to genetic diversity [13] specially when the initial population density is small, which if weakened may result in reducing the fitness and fecundity of the grizzly bears, and a larger requirement for  $MVP$  may require introducing new individuals into the existing population for avoiding “genetic meltdown” that

eventually may lead to extinction of the grizzly population in the monitored area.

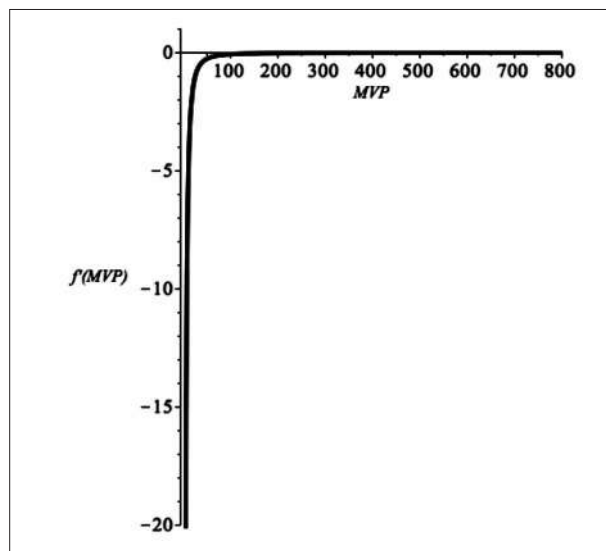
–  $SH$  regulates accordingly the value of  $MVP$ ; higher  $SH$  (better quality habitat for grizzly bears) induces lower requirement for  $MVP$ . The latter two tables above and the Figures 7 & 8 show clearly this phenomenon.

### Conclusion

As mentioned in [5], “Estimating minimum viable population and reserve size is a fundamental cornerstone of conservation biol-



**Fig. 6.**  $f'(MVP) \neq 0$  on the interval  $U = (0, 811)$ . The time unit considered was  $t = 1$ , and the following values were considered for parameters:  $K = 1493$ ,  $r = 1.04$ ,  $SH = 0.433$ ,  $P_0 = 353$ . The value of  $r$  was taken from [9], the values of  $P_0$  and  $SH$  were taken from [12], and we used our estimate that we found for  $K = Kp$



**Fig. 7.** The following values were considered for parameters:  $K = 1493$ ,  $r = 1.04$ ,  $SH = 0.433$ ,  $P_0 = 353$ . The value of  $r$  was taken from [9], the values of  $P_0$  and  $SH$  were taken from [12], and we used our estimate that we found for  $K = Kp$

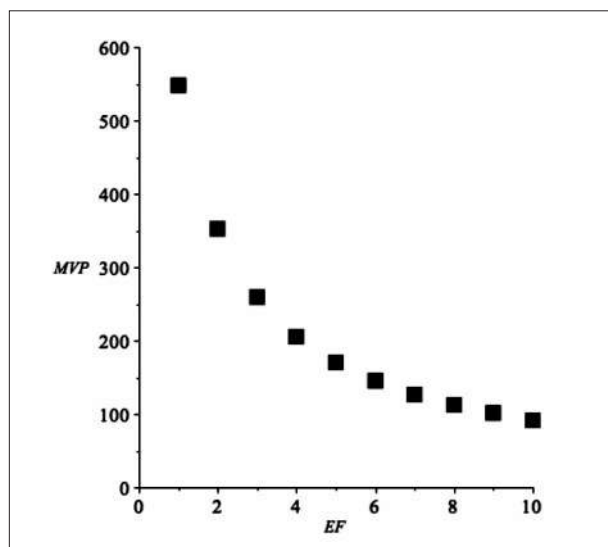


Fig. 8. The following values were considered for parameters:  $K = 1493$ ,  $r = 1.04$ ,  $SH = 0.433$ ,  $P_0 = 353$ . The value of  $r$  was taken from [9], the values of  $P_0$  and  $SH$  were taken from [12], and we used our estimate that we found for  $K = K_p$

ogy”, we bring a new approach on estimating the minimum viable population. Our approach is by correlating the minimum viable population with the fitness of the population and the Safe Harbour ( $SH$ ), the new measuring function introduced by the Government of Alberta. Studies on the estimation of the minimum viable population correlated with the fitness of a population have been performed on different species in botany and zoology, as for example in [14–16]. Our analytical model has the advantage of relying on a small number of parameters, and it can be enhanced by adding new ones, such as the coefficient of inbreeding in the species, or by taking into account other factors that may influence the dynamics of the grizzly bear population. Yet, a viable analysis of the population involves a substantially greater number of parameters that must be taken into account, and many of these parameters cannot be feasibly determined [4].

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